"The 'kTC Noise' – Equipartition, Fluctuation-Dissipation and Detection Sensitivity"

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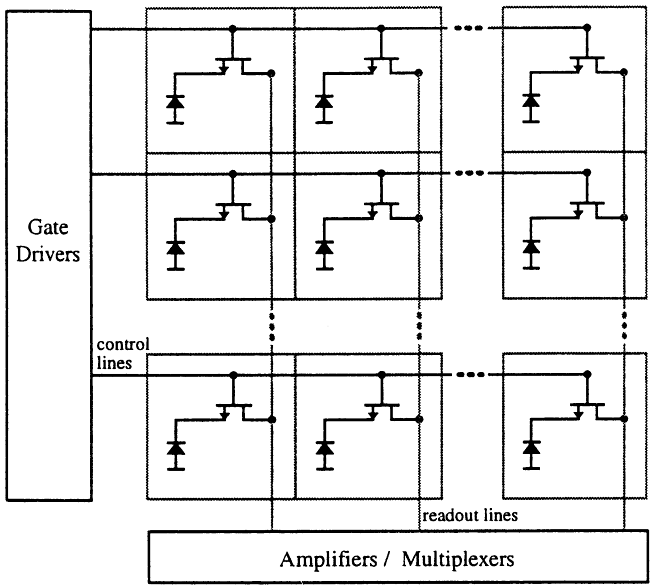
(Seminar at BNL on June 5, 2002)

Noise in electronic and electromechanical systems is usually analyzed in the context of its frequency spectrum and of all sensor parameters. The so called "*kTC*" noise, i.e., the fluctuation of the charge stored on a capacitance C, sometimes referred to as "reset noise", and/or, "reset transistor noise", refers to the *total fluctuation* as governed by the equipartition principle. In some cases, it represents a limit to the measurement accuracy. In some other cases, with most radiation detectors and CCDs, it only represents the worst case, and a much better noise performance can be achieved. While the total charge fluctuation on a capacitance is determined as *kTC* by the equipartition principle, the noise affecting the measurement can be reduced to a very small fraction of this value, by minimizing the dissipative real part of the node impedance, and by restricting the bandwidth to the region of the frequency spectrum where the noise from the dissipative part is negligible. The total fluctuation limit is most difficult to avoid in the simplest detector readout schemes, such as with pixel matrix imagers. The relation between the equipartition principle and the dissipation-fluctuation theorem on the one side, and the detailed properties of the noise and circuit functions on the other, is explored. The equivalence between the electronic circuits and electromechanical systems will be discussed. In particular, the following questions will be addressed:

- Spectrum and autocorrelation function of the *kTC* noise;
- Transient behavior of *kTC* noise in switched and sampled circuits;
- If *kTC* on 1pF is 400 rms e, how can one achieve <10 rms e?
- The noise in pixel matrix imagers: When can the *kTC* noise be reduced?
- Active reset in pixel matrix imagers and the role of feedback;
- *kTC* noise limits in single-electron electronics;
- Equipartition, and the noise in mechanical sensor systems (e.g., MEMS cantilevers, vibration insulators in gravity wave detectors).
- Noise in resonant (electromechanical) systems and the role of the quality factor *Q*.

$$\boldsymbol{s}_q^2 = k_B T C$$

Is kTC a limit?



a) Matrix imager: noise?

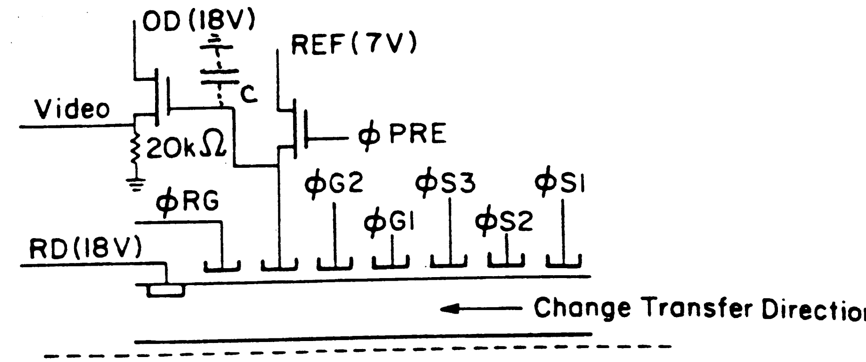
b) CCD

$C \lesssim 50\,fF$

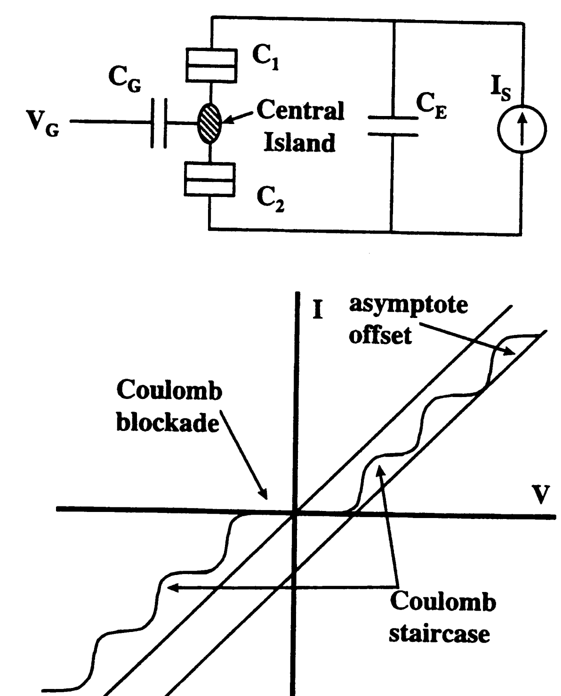
$\sigma_q \approx 80$ rms e

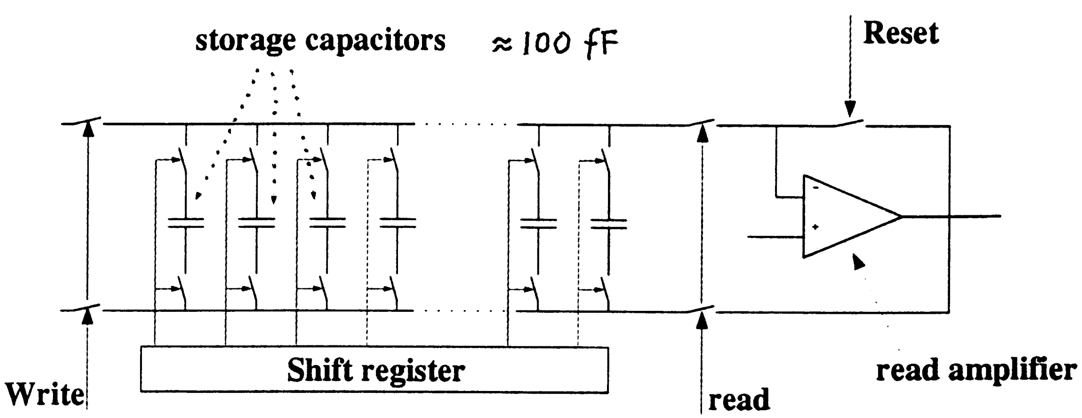
$\downarrow$

< 1 e



c) single e response





c) Analog memory: dynamic range?

Mechanical systems

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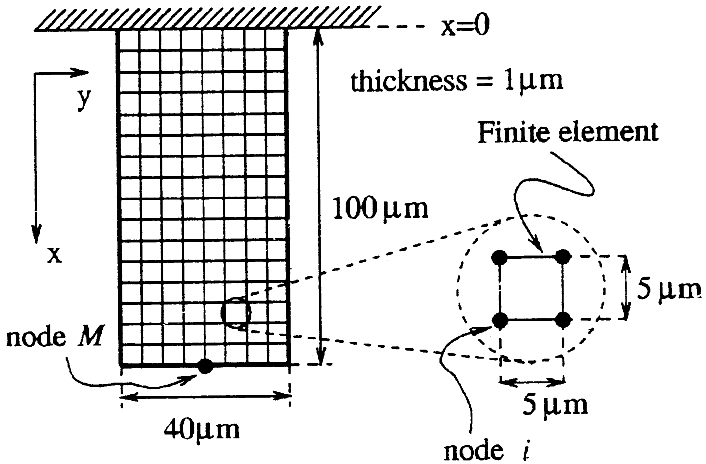
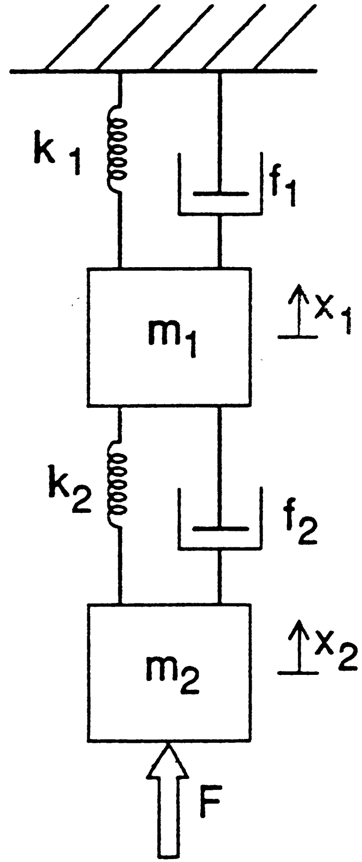


FIG. 3. FEM modeling of a cantilever beam and the electrical model. The length and the width of the cantilever is 100 and 40 μm, respectively. The thickness of the cantilever is 1 μm. Cantilever material is silicon (Young modulus, $E = 130$ Pa, density, $\rho = 2.332$ g/cm^3, Poisson ratio, $\sigma = 0.278$). Node M is in the middle of the free end.



Sensors : AFM, acceleration, pressure, acoustics

Vibration isolators (e.g., in gravitational detectors)

Equipartition Theorem ½k_BT per:

"degree of freedom"
"energy storage mode"
"quadratic term"
"state variable"

E

$$\frac{1}{2}C\overline{v^2} = \frac{1}{2}k_B T$$

$$\frac{1}{2}L\overline{i^2}$$

$$\frac{1}{2}m\overline{v_x^2}$$

$$\frac{1}{2}I\overline{\varphi'^2}$$

$$\frac{1}{2}k\overline{x^2}$$

capacitance	C
inductance	L
mass	m
moment of inertia	I
spring	k

If $P(E) \propto \exp(-E/kT)$
(<u>Boltzmann</u>),
then probability dP for $v, v+dv$

$$dP = A_0 \exp\left(-\frac{1/2Cv^2}{k_B T}\right)dv$$

from $\int dP = 1$,

$$A_0 = \frac{C}{2\pi k_B T}$$, and the variance is,

$$\sigma_v^2 = \overline{v^2} = A_0 \int_{-\infty}^{\infty} v^2 \exp\left(-\frac{Cv^2}{2k_B T}\right)dv =$$

$$\sigma_v^2 = \frac{k_B T}{C}$$

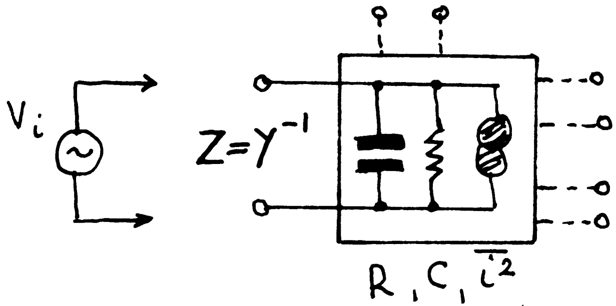
$$\sigma_q^2 = \sigma_v^2 \cdot C^2 = k_B T C$$

<u>total fluctuation</u>
(for entire frequency spectrum)

Fluctuation-Dissipation Theorem

Key concept: the ***driving point*** impedance/admittance

$$Z = Y^{-1}$$



Real part determines ***dissipation***:

$$Y = \frac{1}{R} + j\omega C$$

$$\mathrm{Re}\{Y\} = \frac{1}{R}$$

$$P_d = V_i^2 / R$$

. . . . and ***fluctuation*** (Nyquist):

$$\overline{i^2} = 4k_B T \quad \mathrm{Re}\{Y\} = 4k_B T \frac{1}{R} \quad [\mathrm{A}^2/\mathrm{Hz}]$$

n – port network: Y or Z – parameter network matrix: (Y_{11} to Y_{nn})

Then:

$$\overline{e_{ii}^2} = 4k_B T \quad \mathrm{Re}\{Z_{ii}\} \quad [\mathrm{V}^2/\mathrm{Hz}]$$

$$\overline{i_{ii}^2} = 4k_B T \quad \mathrm{Re}\{Y_{ii}\} \quad [\mathrm{A}^2/\mathrm{Hz}]$$

Correlation port-to-port: $\overline{e_i \cdot e_k^*} = 4k_B T \,\mathrm{Re}\{Z_k^i\}$

Brownian Motion	***Energy & Particle Distribution, Equipartition, Entropy, etc.***	***Noise***	***Fluctuation -Dissipation***	***CCDs, Switched Capacitor Circuits; Single - e Transistors; MEMS Sensors, AFM, Gravitational Wave Detectors***
Brown (1827)	Maxwell (1860)	Johnson (1926)	Callen,	McCombie (1952)
Einstein (1905-7)	Boltzmann (1871-8)	Nyquist (1928)	Welton,	Saulson (1990)
Ornstein,	Einstein (1905)		Greene (1951-2)	Gabrielson (1993)
Uhlenbeck,	Nernst			Yaralioglu-Atalar (1999)
Goudsmit (1929-30)	Onsager (1931)			
Langevin				

Tools:

- statistical mechanics
- circuit analysis
- probability & statistical analysis of signals

Circuit & Signal Analysis:

Campbell,
Norton,
Thevenin,
Bode,
Wiener,
Khintchine,
Parseval

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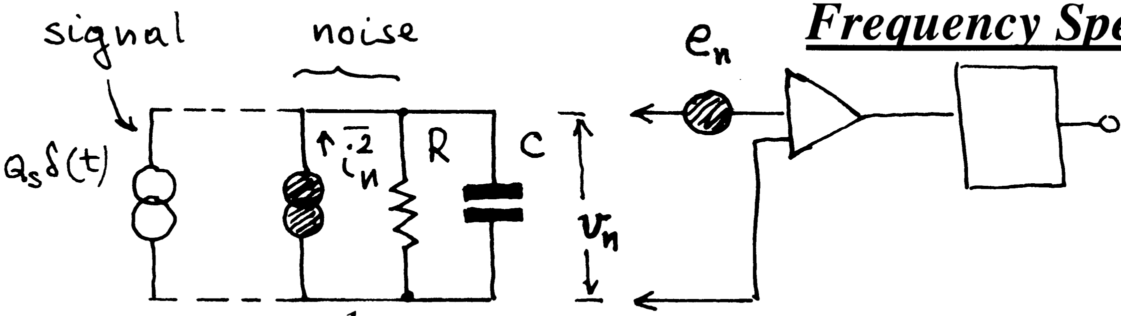
Bo Yu

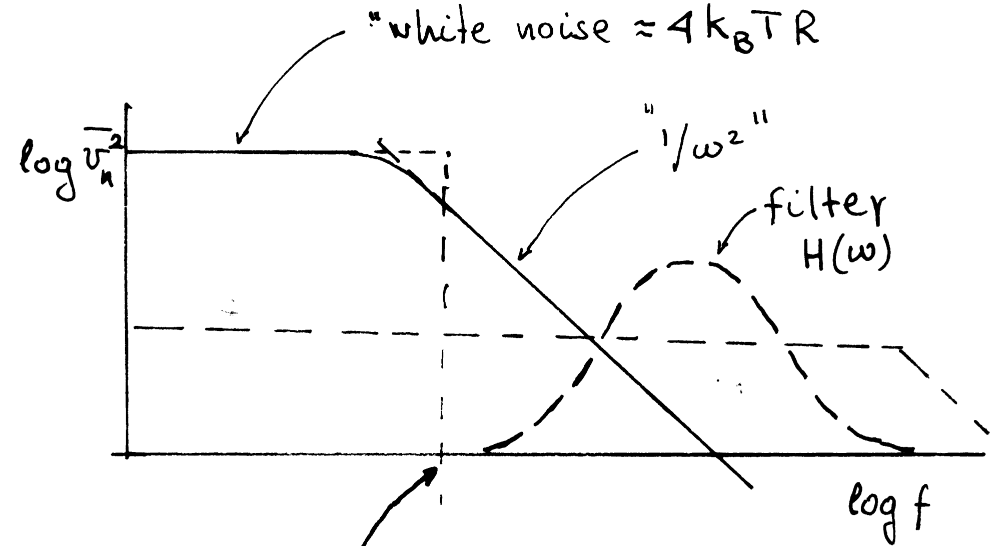
Frequency Spectrum



$$Y = \frac{1}{R} + j\omega C$$

$$\overline{i_n^2} = 4k_B T / R$$

$$\overline{v_n^2} = \overline{i_n^2}\left|Y^{-1}(\omega)\right|^2 = 4k_B TR \frac{1}{1+(\omega CR)^2}$$



Eq. bandwidth $f_h = \dfrac{1}{4RC}$

Total fluctuation:

$$\sigma_v^2 = \int_o^\infty \overline{v_n^2}\, df = \frac{k_B T}{C}$$

$$\sigma_q^2 = k_B TC$$

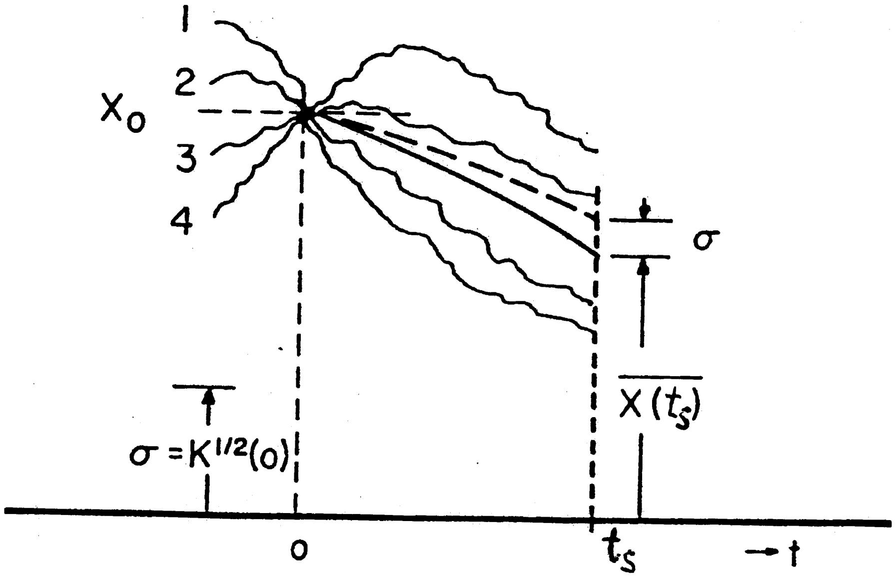
"independent of R"

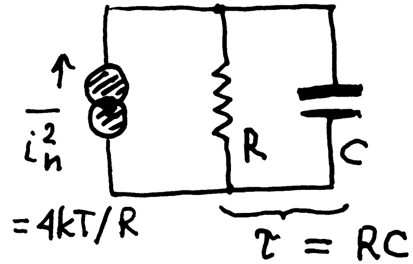
at 293k:

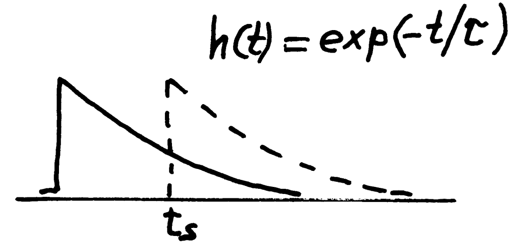
C	***σ$_q$*[rms e]**	***σ$_v$***
1 pf	400	63µV
100 fF	127	200µV
10 fF	40	0.63mV
1aF	0.4	63mV

1. C=1pF, R=1GΩ $\}$ RC = 1ms, f_h = 250 Hz
2. C=100fF, R=10kΩ $\}$ RC=1ns, f_n = 250 MHz

Autocorrelation Function

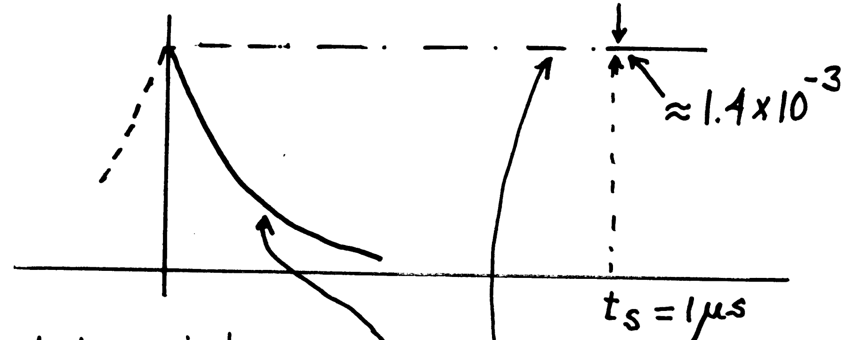






$$k(t_s) = \int_0^{\infty} h(t+t_s)\cdot h(t)\,dt = \exp\left(-\frac{|t_s|}{\tau_F}\right)$$

$$\frac{\overline{X(t_s)}}{X_0} = \frac{K(t_s)}{K(0)} = k(t_s)$$



$$\sigma^2 = K(0) - \frac{K^2(t_s)}{K(0)} = \sigma^2\left[1 - k^2(t_s)\right]$$

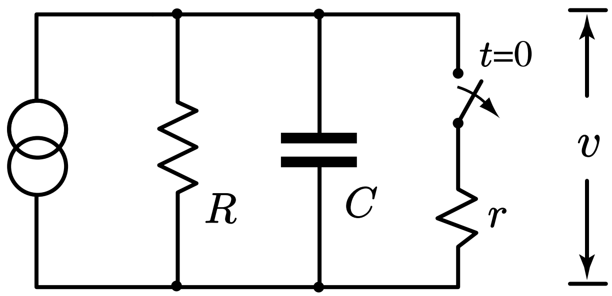
Reset transistor:

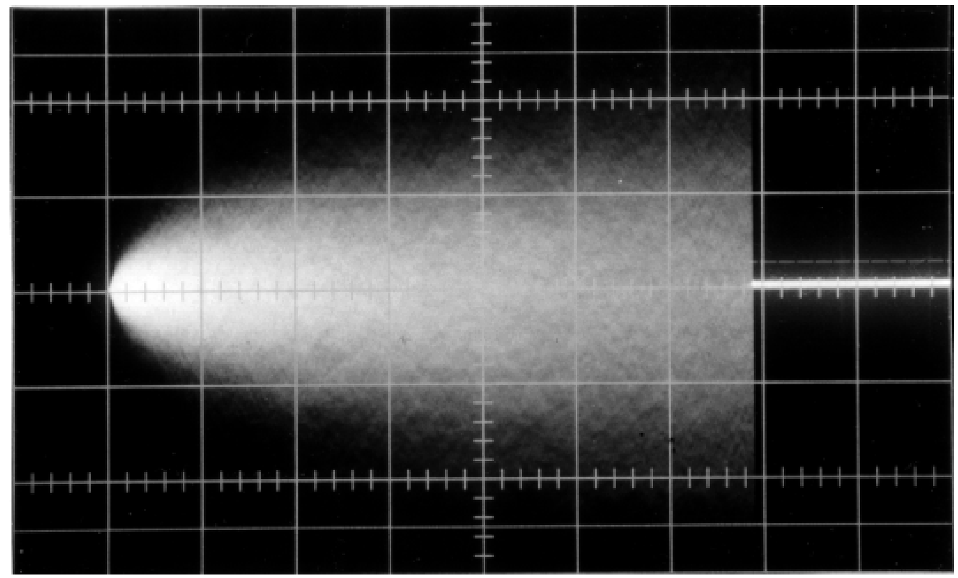
$$\left.\begin{array}{l} C = 100\,fF \\ R_{ON} = 10^6\,\Omega \end{array}\right\}\ 100\ ns \qquad \left\{\begin{array}{l} C = 100\,fF \\ R_{OFF} = 10^{13}\,\Omega \end{array}\right\}\ 1s$$

$$\frac{\sigma_v}{(k_B T/C)^{1/2}} = \left(1 - e^{-2t_s/\tau}\right)^{1/2}$$

σ_v, σ_q vs. Time — Transient Response

$$\overline{i_n^{\,2}} = 4k_BT/R$$



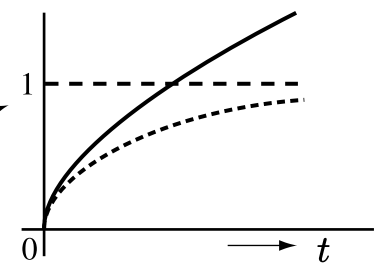


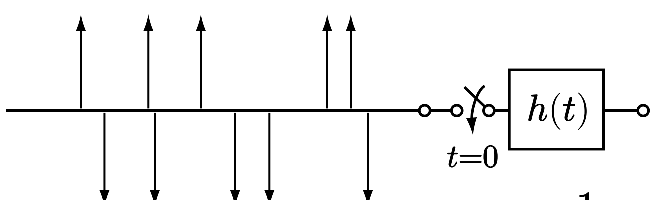
Campbell:

$$\sigma_v^2 = \frac{1}{2}\,\overline{i_n^2}\int_0^t h^2(u)du = \frac{k_BT}{C}(1-e^{-2t/\tau})$$

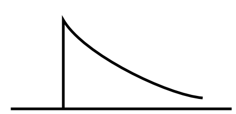
$$\underline{\sigma_v/(k_BT/C)^{1/2}} = \underline{(1-e^{-2t/\tau})^{1/2}}$$

$$\frac{\sigma\,(t)}{(k_BT/C)^{1/2}}$$





$$h(t) = \frac{1}{C}\,e^{-t/\tau}$$

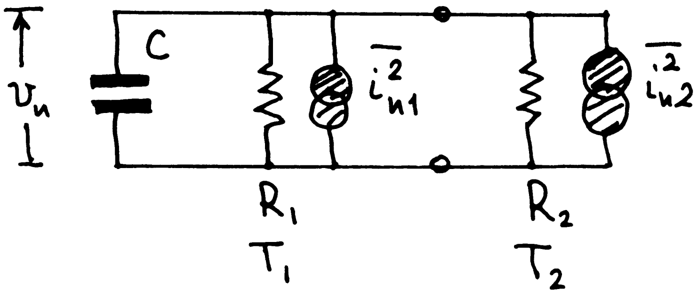


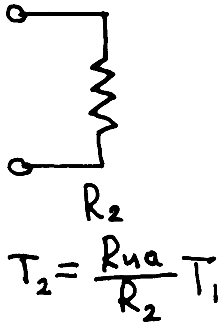
For $t/\tau \ll 1$: $\quad \sigma_v^2 \propto t$ (random walk)

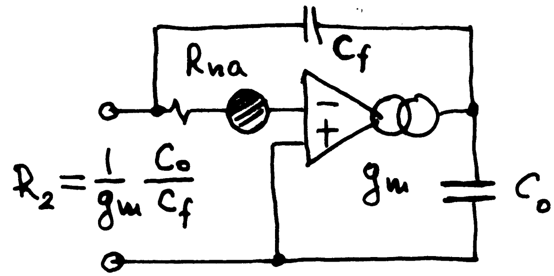
$$\sigma_v \approx (k_BT/C)^{1/2}\cdot(2\frac{t}{\tau})^{1/2}$$

For $\left.\begin{aligned}\tau = RC &= 1\text{s},\\ t &= 1\mu\text{s}\end{aligned}\right\}$ $\sigma_v/(k_BT/C)^{1/2} \approx 1.4\times10^{-3}$

Two Resistors and "Cooled Damping"







Total fluctuation (all frequencies):

$$\sigma_v^2 = \frac{k_B}{C} \cdot R_1 \| R_2 \left(\frac{T_1}{R_1} + \frac{T_2}{R_2} \right)$$

1. for $T_1 = T_2 \rightarrow$ $\sigma_v^2 = \frac{k_B T_1}{C}$

2. if $T_2 = 0$ $\rightarrow$ $\sigma_v^2 = \frac{k_B T_1}{C} \cdot \frac{1}{1 + R_1 / R_2}$

With feedback (cooled) damping:

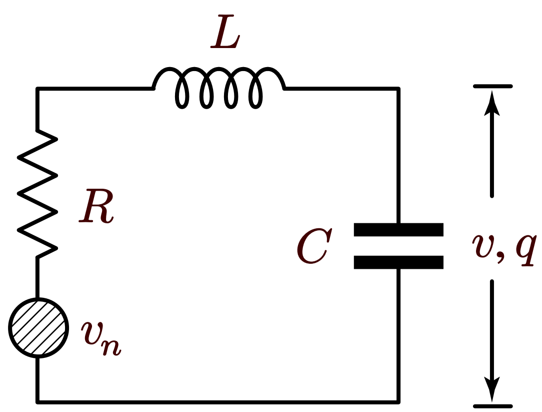
$$\sigma_v^2 = \frac{k_B T_1}{C} \cdot \frac{1}{1 + R_1 / R_2} \left(1 + \frac{R_{na} R_1}{R_2^2} \right)$$

σ_v min for $R_2 = (R_1 R_{na})^{1/2}$

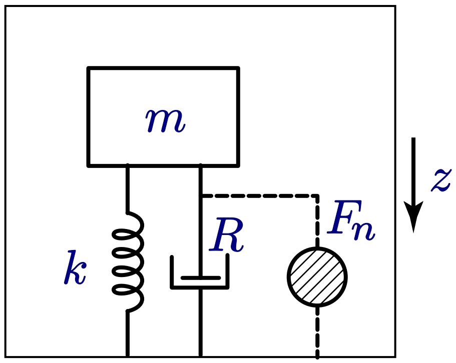
$$\rightarrow \sigma_v^2 = \frac{kT_1}{C} \cdot \frac{2}{1 + (R_1 / R_{na})^{1/2}}$$

$$\left. \begin{array}{l} R_1 = 10^6 \Omega \\ R_{na} = 100 \Omega \end{array} \right\} \begin{array}{l} R_2 = 10^4 \Omega \\ \sigma_v^2 \cong \frac{k_B T_1}{C} \cdot \frac{1}{50} \end{array}$$

Resonant Systems



$$\boldsymbol{V} \longleftrightarrow \boldsymbol{F}$$
$$\boldsymbol{q} \longleftrightarrow \boldsymbol{z}$$
$$\boldsymbol{i} \longleftrightarrow \boldsymbol{v_z}$$
$$\boldsymbol{L} \longleftrightarrow \boldsymbol{m,\ I}$$
$$\boldsymbol{C} \longleftrightarrow \boldsymbol{1/k}$$



$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i dt = v_n$$

$$\frac{d^2q}{dt^2} + \frac{\omega_0}{Q}\frac{dq}{dt} + \omega_0^2 q = v_n C \omega_0^2$$

$$\overline{q_n^2} = 4k_B T R C^2 \cdot |G(f)|^2$$

$$m\frac{d^2z}{dt^2} + R\frac{dz}{dt} + kz = F_n$$

$$\frac{d^2z}{dt^2} + \frac{\omega_0}{Q}\frac{dz}{dt} + \omega_0^2 z = F_n\frac{\omega_0^2}{k}$$

$$\overline{z_n^2} = 4k_B T R\frac{1}{k^2}|G(f)|^2$$

$$\boxed{|G(f)|^2 = \frac{1}{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left(\frac{\omega}{\omega_0}\right)^2 \frac{1}{Q^2}}}$$

$$\omega_0 = \sqrt{LC}$$

$$Q = \frac{\sqrt{L/C}}{R} = \frac{\omega_0 L}{R}$$

$$\omega_0 = \sqrt{k/m}$$

$$Q = \omega_0 m/R$$

$$\sigma_q^2 = \int \overline{q_n^2} df = \underline{k_B T C}$$

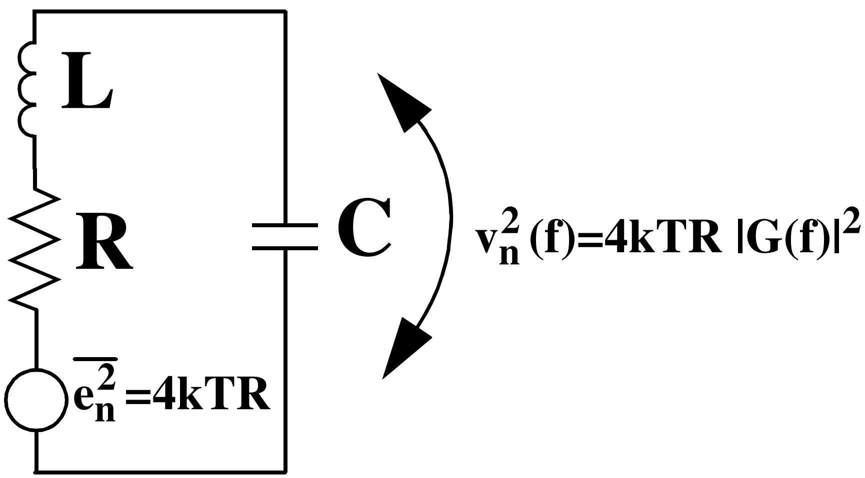
$$\int \overline{z_n^2} df = \underline{k_B T \frac{1}{k}} = \sigma_z^2$$

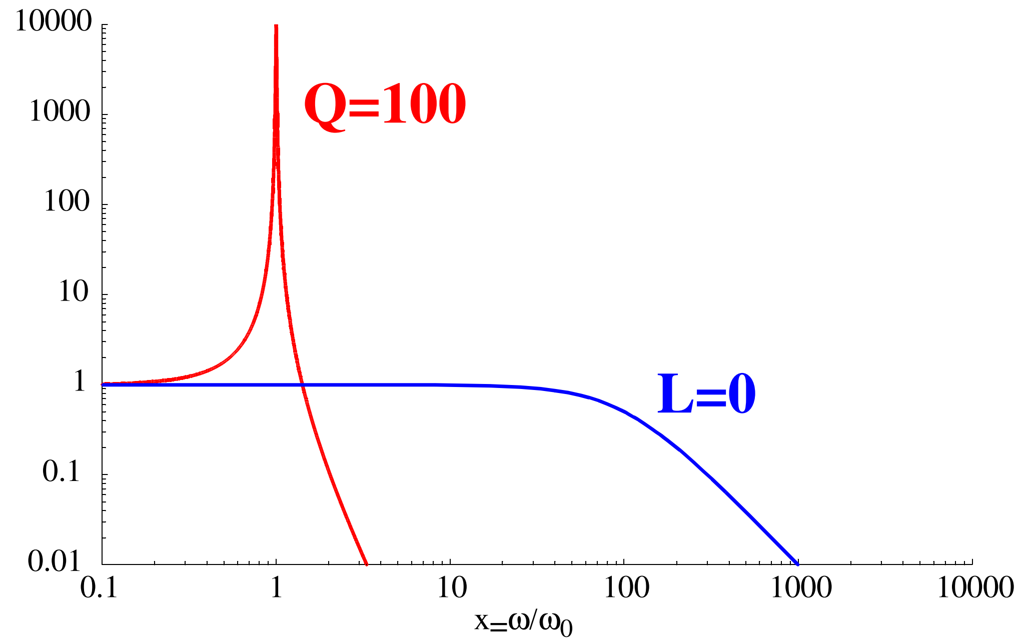
<u>For $\omega/\omega_0 \ll 1$:</u>

$$\overline{q_n^2} = k_B T C \frac{4}{Q\omega_0}$$

$$\overline{z_n^2} = k_B T \frac{1}{k} \cdot \frac{4}{Q\omega_0}$$

Noise Spectral Density and Equipartition Theorem

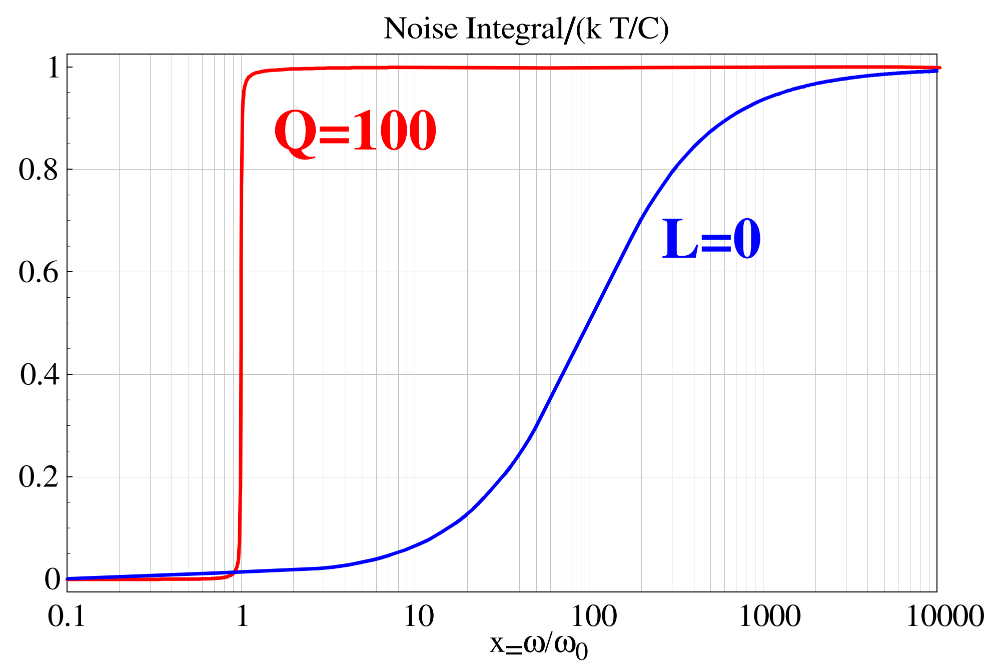




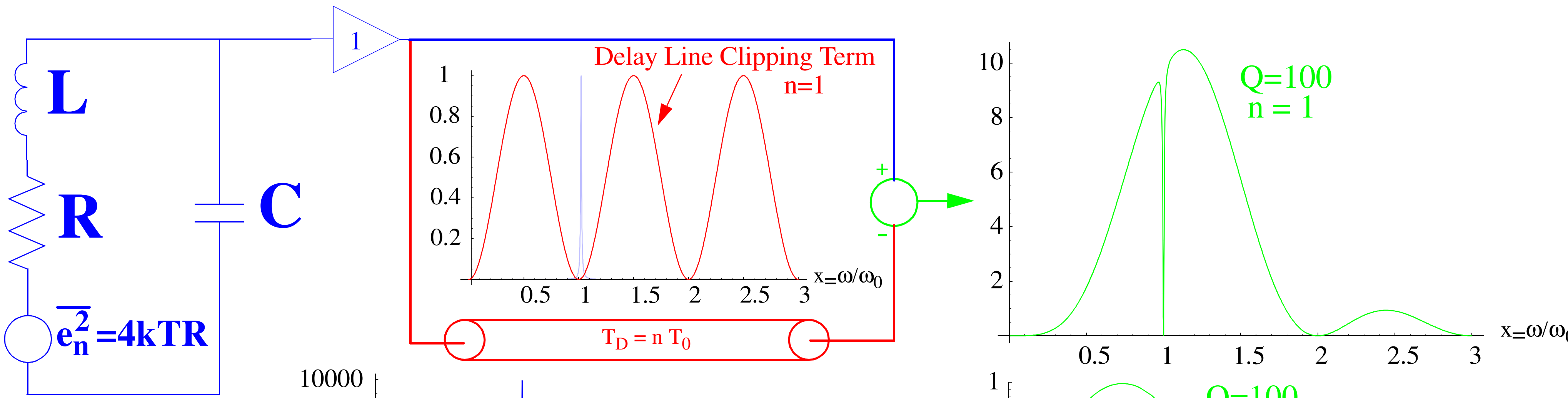
$$|G(f)|^2 = \frac{1}{(1+x^2)^2 + \frac{1}{Q^2}x^2}$$

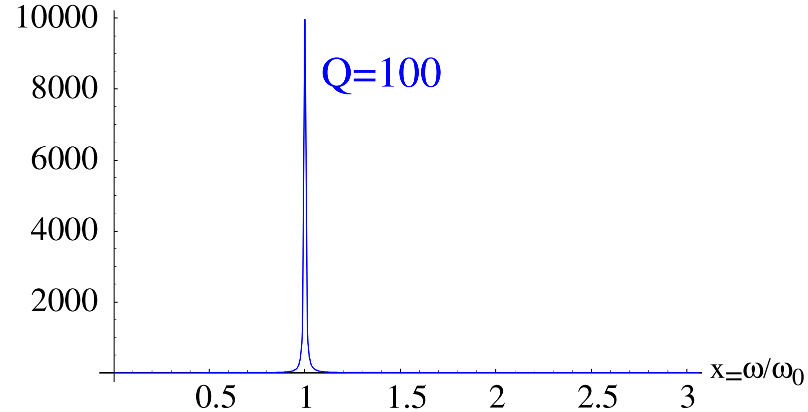
$$|G(f)|^2_{L=0} = \frac{1}{1+\frac{x^2}{Q^2}}$$

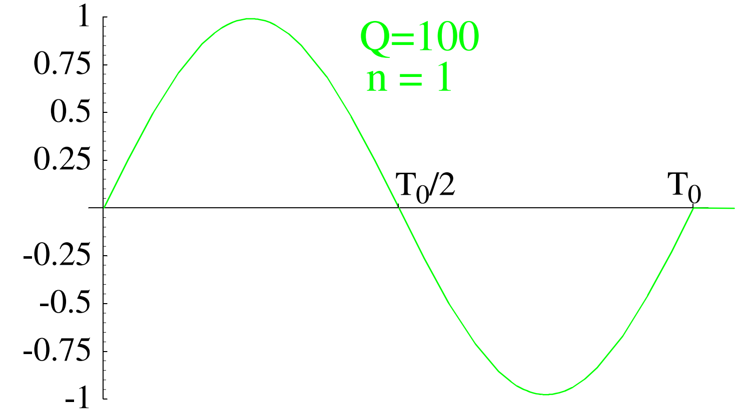
$$4kTR\int_0^\infty |G(f)|^2 df = \frac{kT}{C}$$



Delay Line Noise Cancellation for a High Q Resonant Circuit



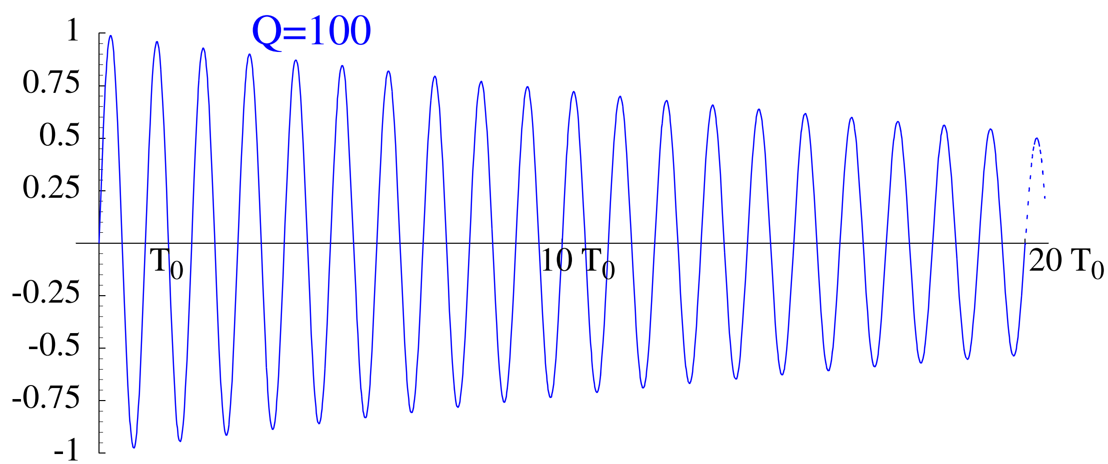




$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$T_0 = \frac{2\pi}{\omega_0}$$

$$Q = \frac{\sqrt{L/C}}{R}$$

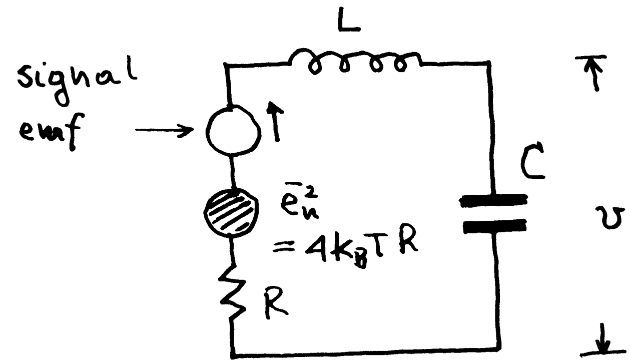


$$\tau = \frac{2Q}{\omega_0} = \frac{Q}{\pi}T_0$$

Impulse Detection by a Resonant Detector

Impulse response:

$$h(t) = \omega_0 \, e^{-\frac{\omega_0 t}{2Q}} \sin \omega_0 t$$



$\omega_0 = (LC)^{-1/2}$

$Q = \dfrac{(L/C)^{1/2}}{R}$

$n =$ number of cycles

Campbell (superposition of noise impulses):

$$\sigma_v^2 = \frac{1}{2} \, 4k_BTR \int_0^t h^2(u)\,du$$

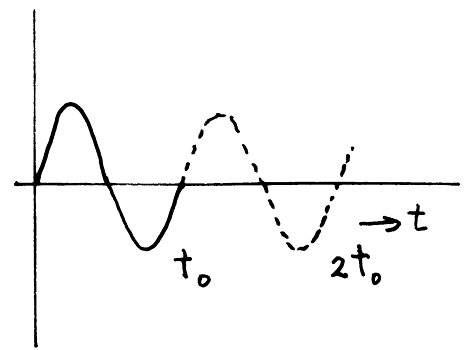
$$\underline{\sigma_v^2(n) = \frac{k_BT}{C}\left(1 - e^{-2\frac{\pi}{Q}n}\right)}$$

for $Q \gg 1$, $n \ll \dfrac{Q}{\pi}$:

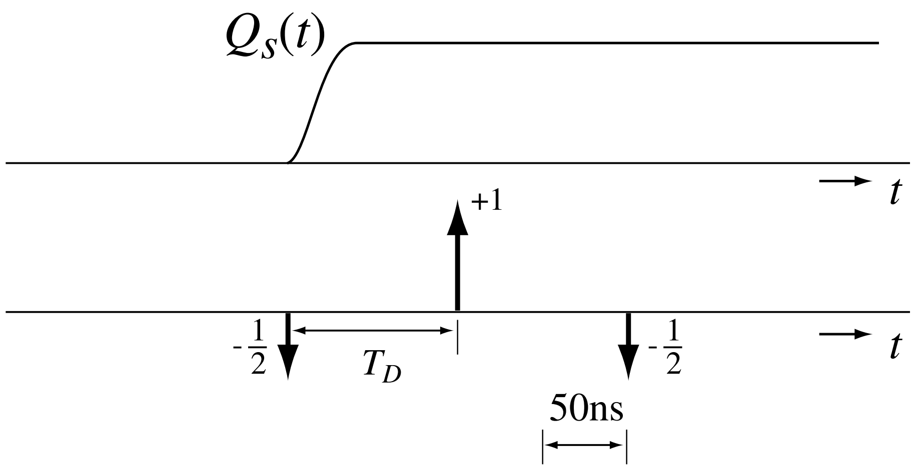
$$\overline{\sigma_v^2}(n) \approx \frac{k_BT}{C} \cdot \frac{2\pi n}{Q}$$

for $\underline{n = 1}$, $Q = 100$

$$\frac{\sigma_v^2(1)}{k_BT/C} \approx \frac{1}{Q/2\pi} \approx \frac{1}{16}$$



Equivalence of Double Delay Line Clipping and Triple Sampling

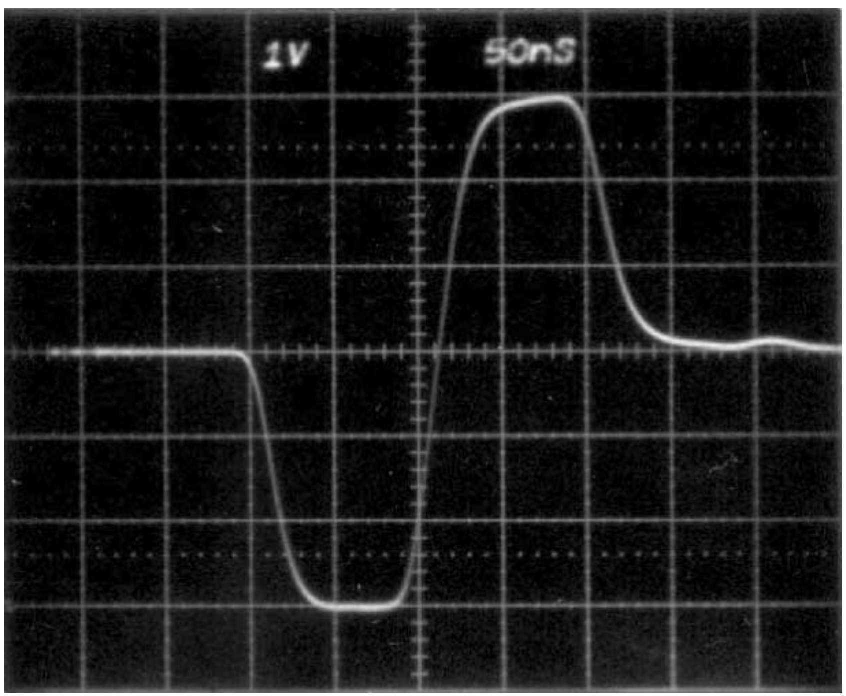


Integrated signal current

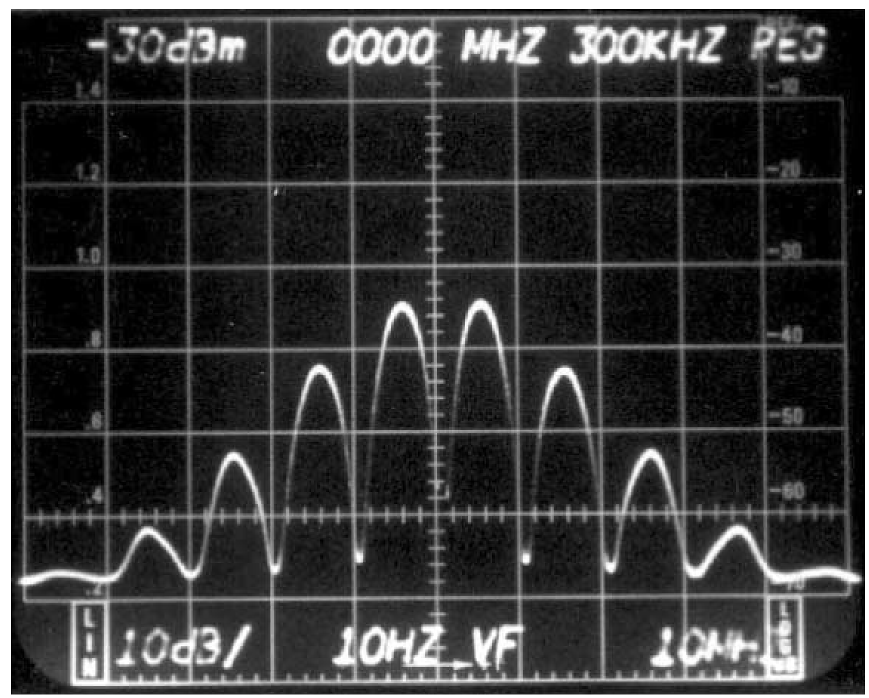
$$\int i_s(t)\, dt$$

$$*$$

"Double delay line clipping"

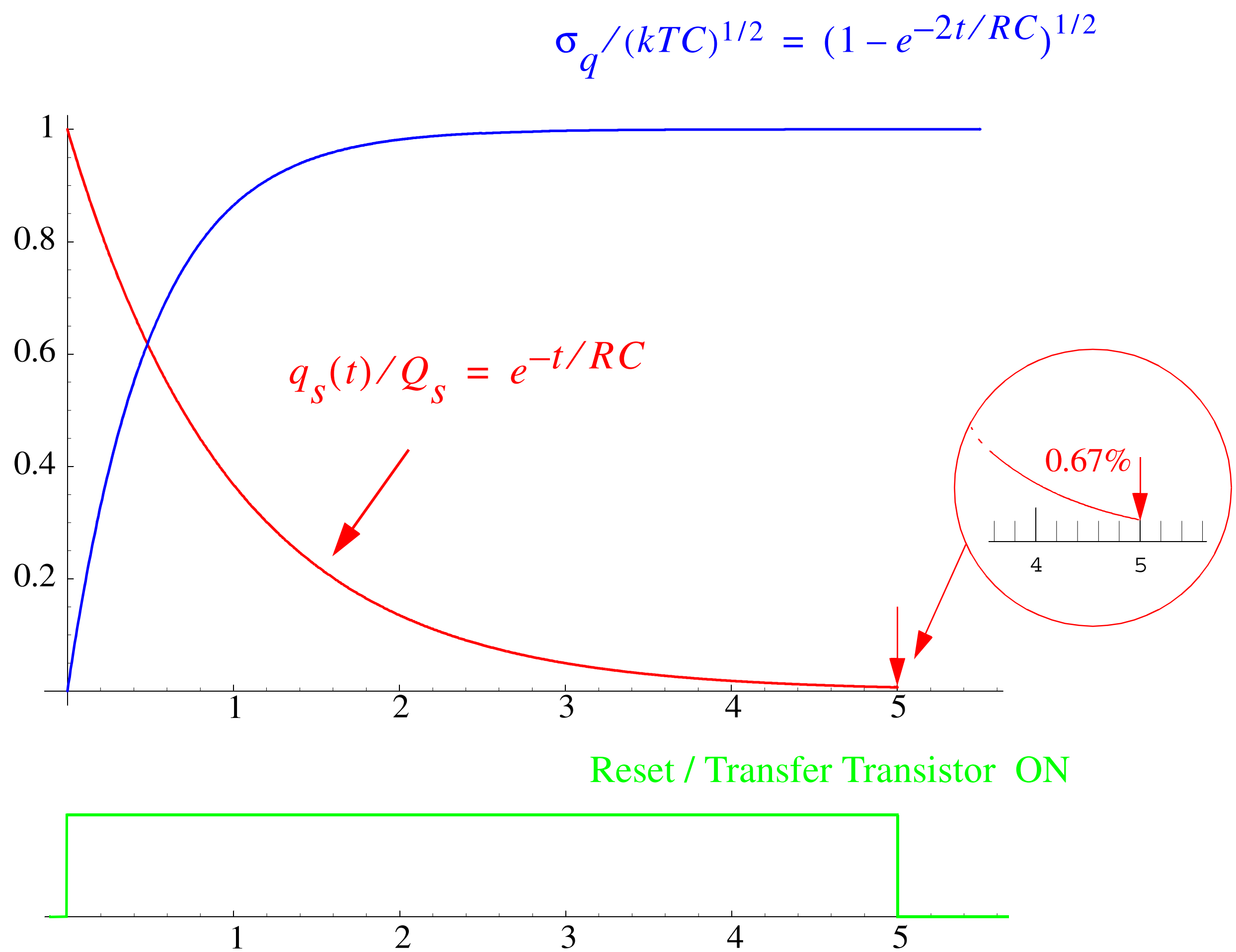


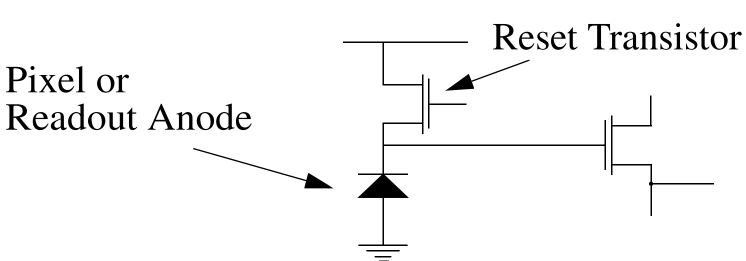
time domain

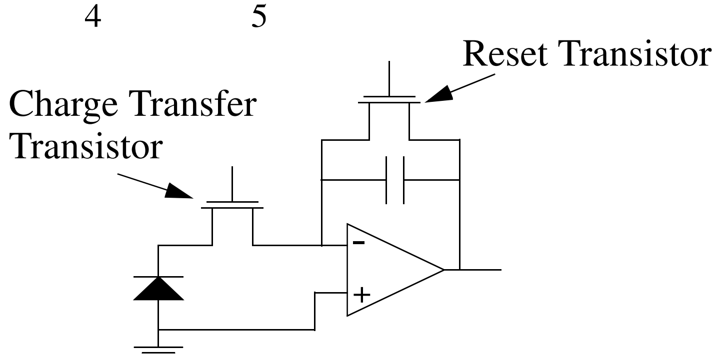


frequency domain

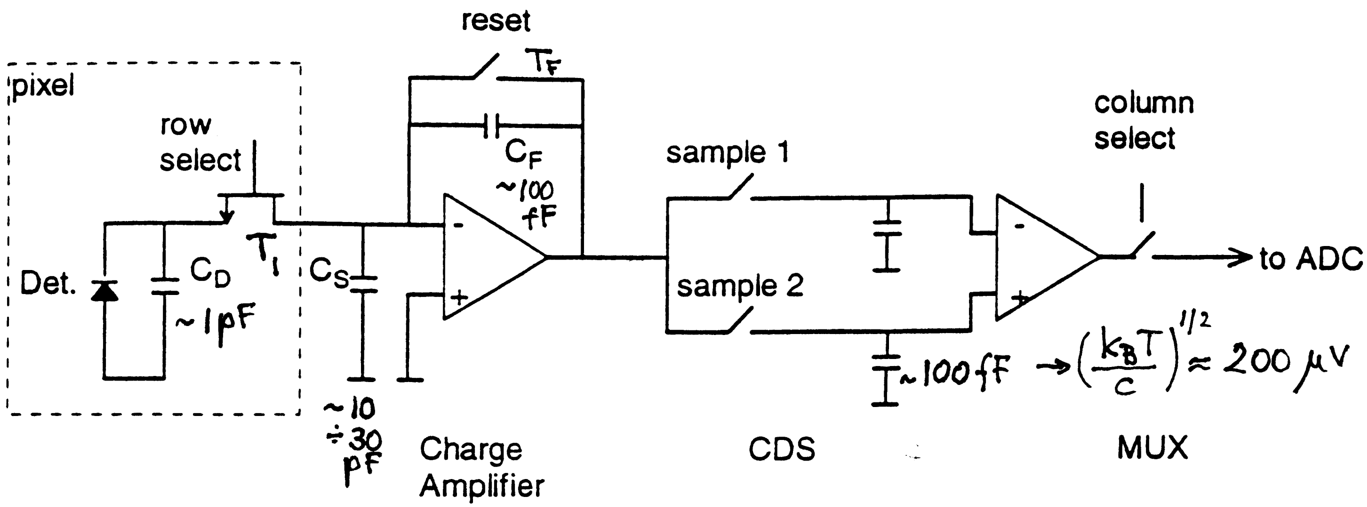
Reset, or Charge Transfer, and Noise

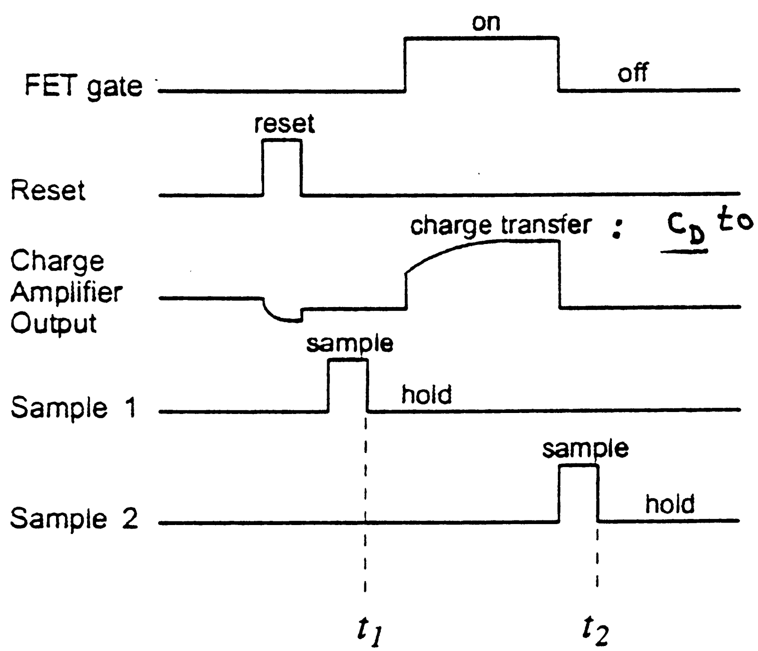






Single Transistor (switch) / pixel Matrix Readout





: $\underline{C_D}$ to $\underline{C_F}$ including $(kTC_D)^{1/2}$

Rapid noise build-up ↑ on C_f

on C_D

$C_F R_{ON} \sim 1\text{-}10$ ns

$C_D R_{ON} \sim 100$ ns $- 1\,\mu s$

$C R_{OFF} \sim 1$ ms $- 1$ s

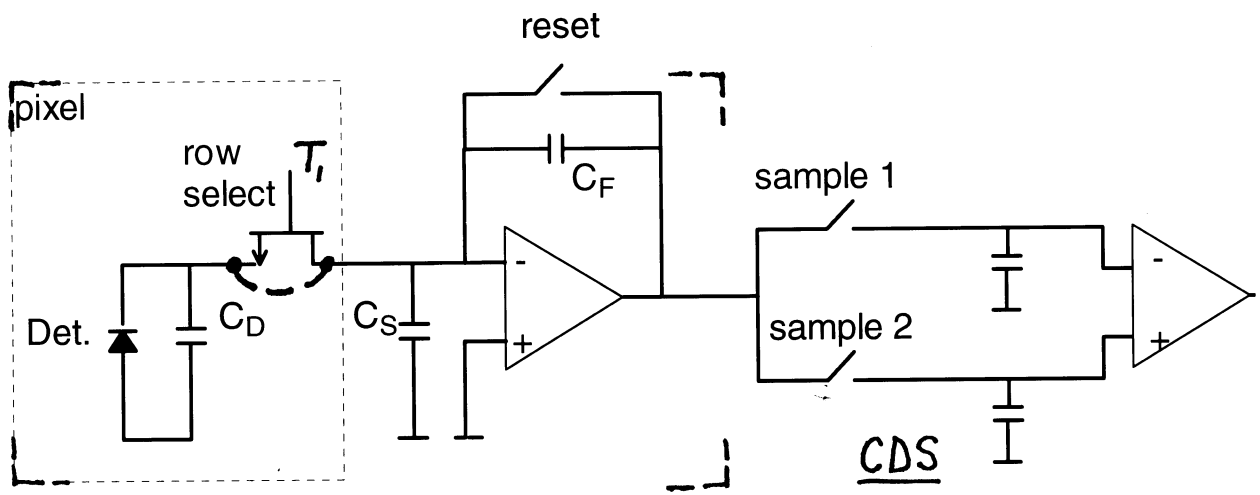
<u>Noise limit</u>:

$(k_B T C_D)^{1/2}$

≈ 400 rms e

for $C_D \approx 1$ pF

Matrix Readout: "Amplified Pixel" with Lowest Noise - Amplifier in the Pixel



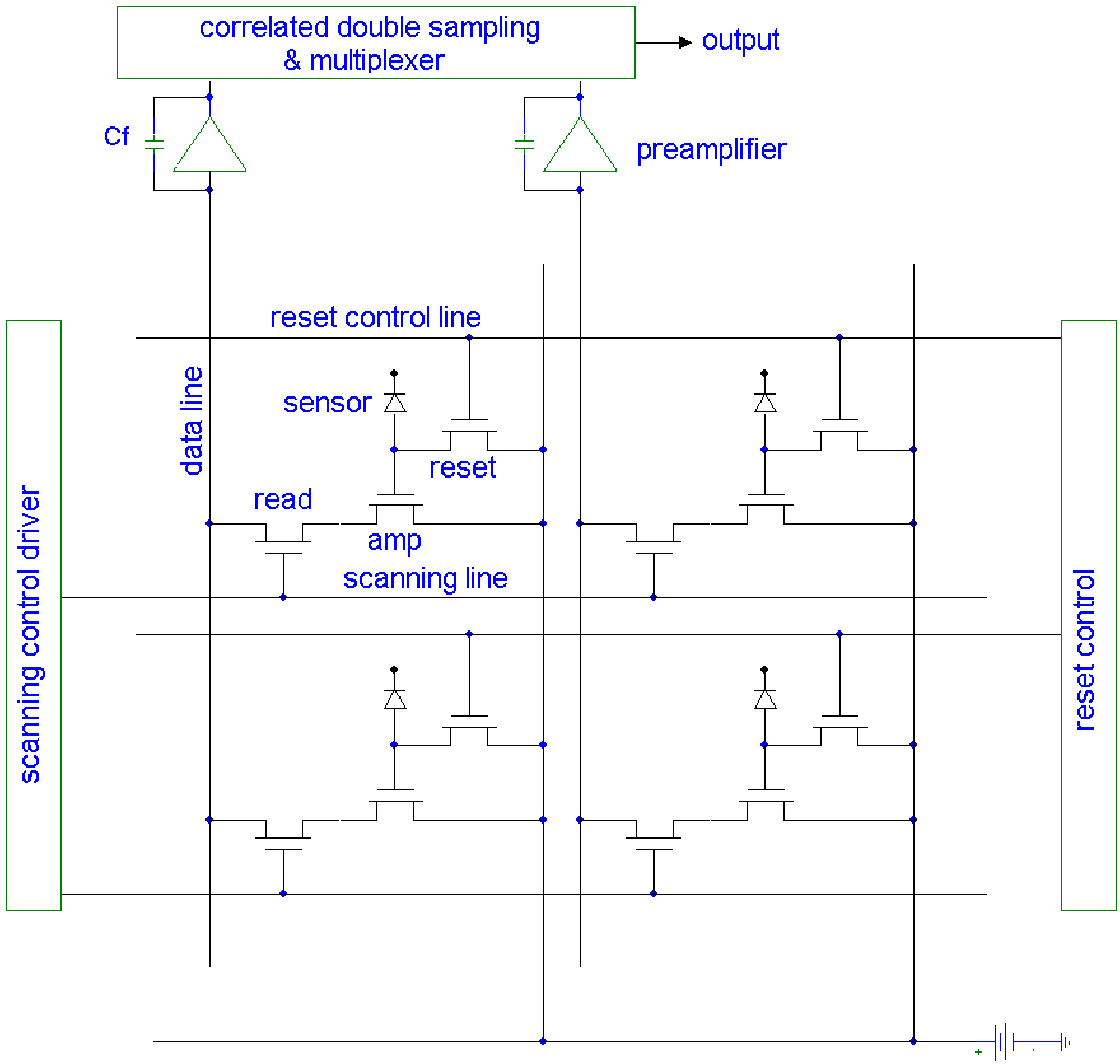
No switched transfer of charge - T_1 out!

Noise arising during reset of C_F:

$\sigma_q = (k_B T C_F)^{1/2}$ *is **lower** than on* C_D, since $C_F \ll C_D$

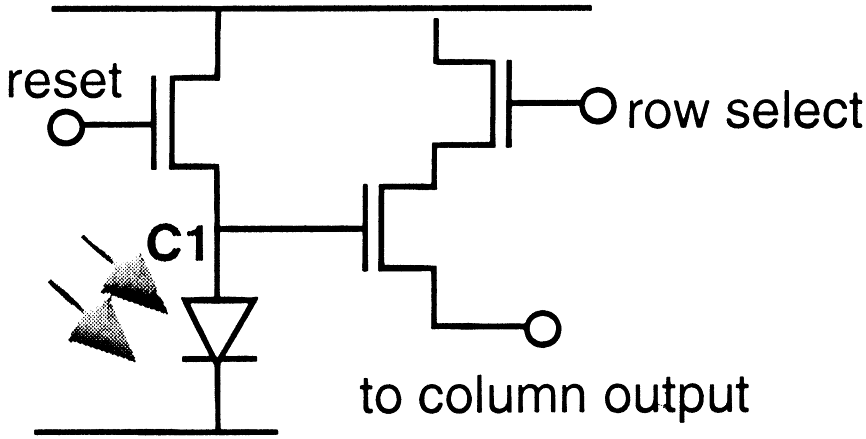
With Correlated Double Sampling (CDS), only amplifier series noise and detector leakage current noise remain.

Matrix readout with “Amplified Pixel”

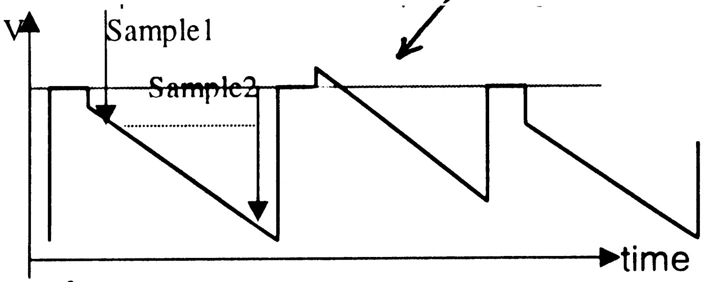


Integrating Detector with Matrix Readout

3T active pixel



integration → $\underline{C_1 R_{OFF} \sim 1s}$

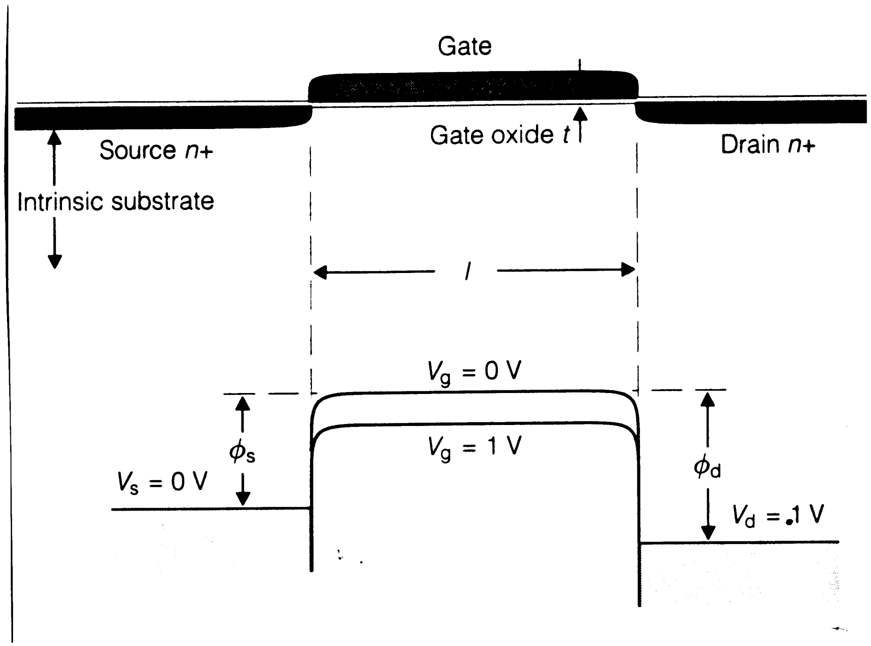


reset

$\underline{C_1 R_{ON}} \sim \underline{1ns - 1\mu s}$

Correlated Double Sampling (CDS)

Can the noise be less than $k_B TC$ during reset or transfer of charge by a CMOS switch?



1. In strong inversion, the channel resistance R_{ON} generates thermal noise:

$$\sigma_q^2 = k_B TC$$

2. In weak inversion, and $V_{DS} > 60\,mV$

$$I_S \propto exp\left(\frac{q_e V_{DS}}{k_B T}\right) \rightarrow g_{DS} = \frac{q_e}{k_B T} I_S$$

$$\overline{i_n^2} = 2q_e I_S = 2k_B T g_{DS} \qquad g_{DS} = 1/R_{ON}$$

$$\sigma_q^2 = \frac{1}{2} k_B TC$$

3. In weak inversion, and $V_{DS} = 0$

$$I_D = -I_S \rightarrow I_{DS} = 0 \ ,$$

but $\overline{i_n^2} = 4k_B T g_{DS}$, and $\sigma_q^2 = k_B TC$

Yes, but with *incomplete charge transfer*, since $V_{DS} \neq 0$
(not in thermal equilibrium)

CCD

$C \gtrsim 50\,fF$

$\sigma_q = (k_B T C)^{1/2} \approx 90$ rms e

After multiple measurements

≈ 0.5 rms e

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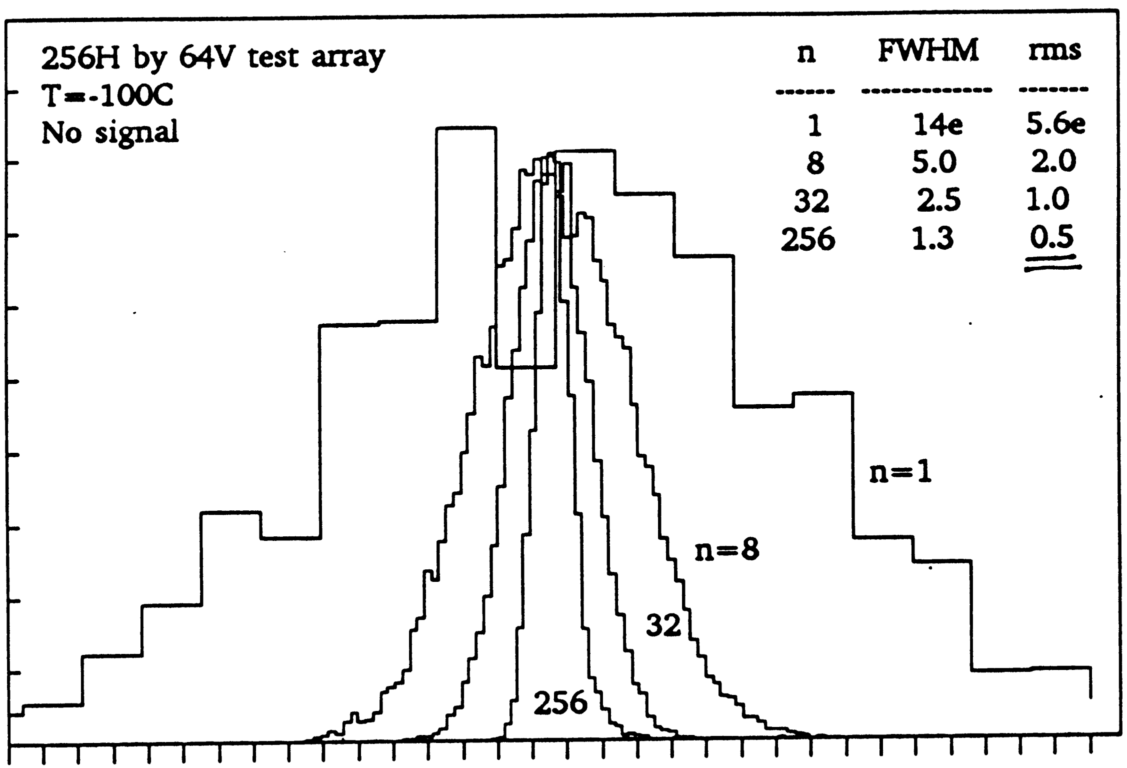
J. Janesick, etal.



X axis = 1 electron per division

Figure 7. Readout noise histograms

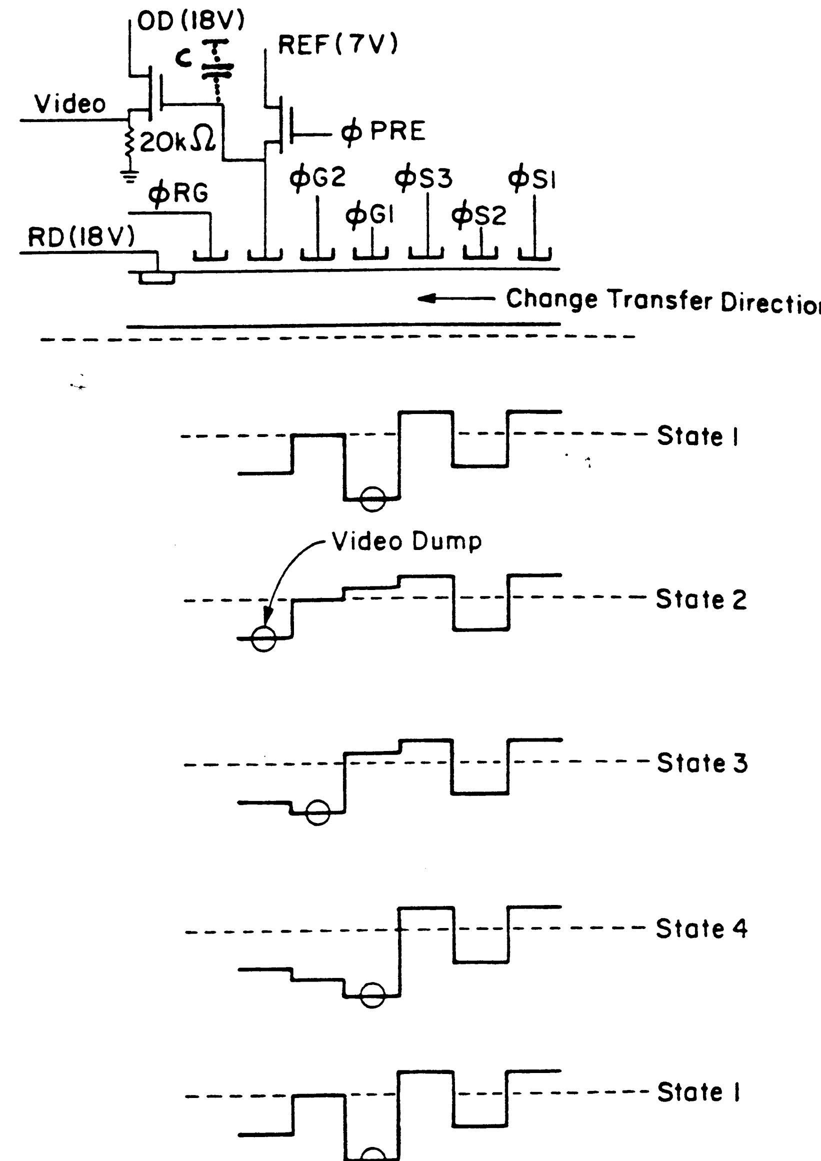
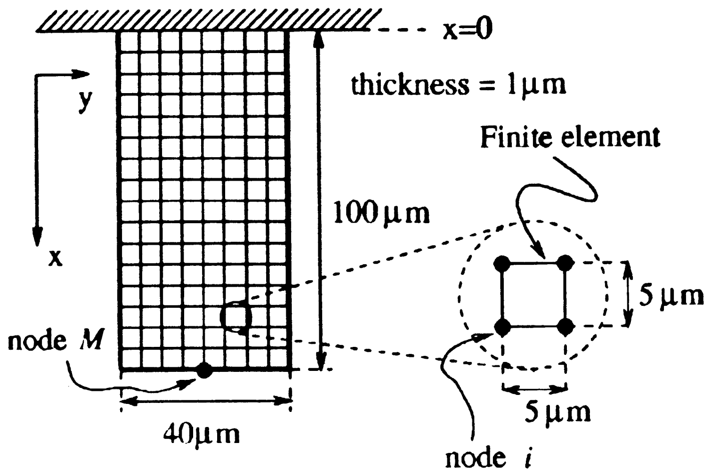


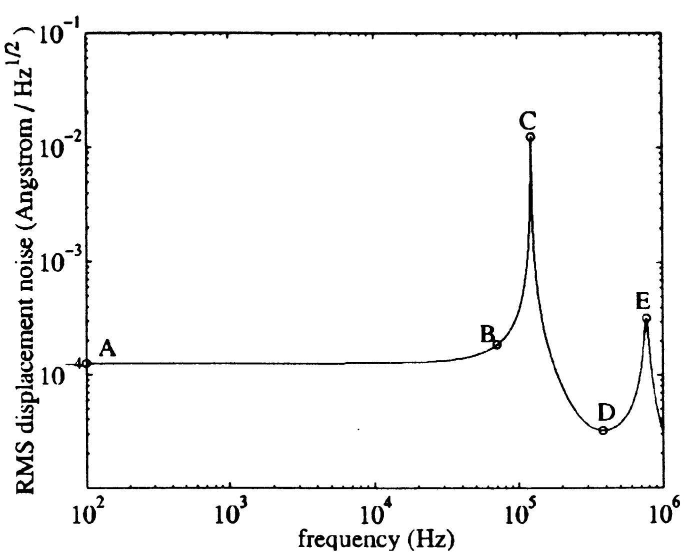
Fig. 2. Schematic diagram of floating gate amplifier and waveform to perform multiple readouts of the same charge packet.

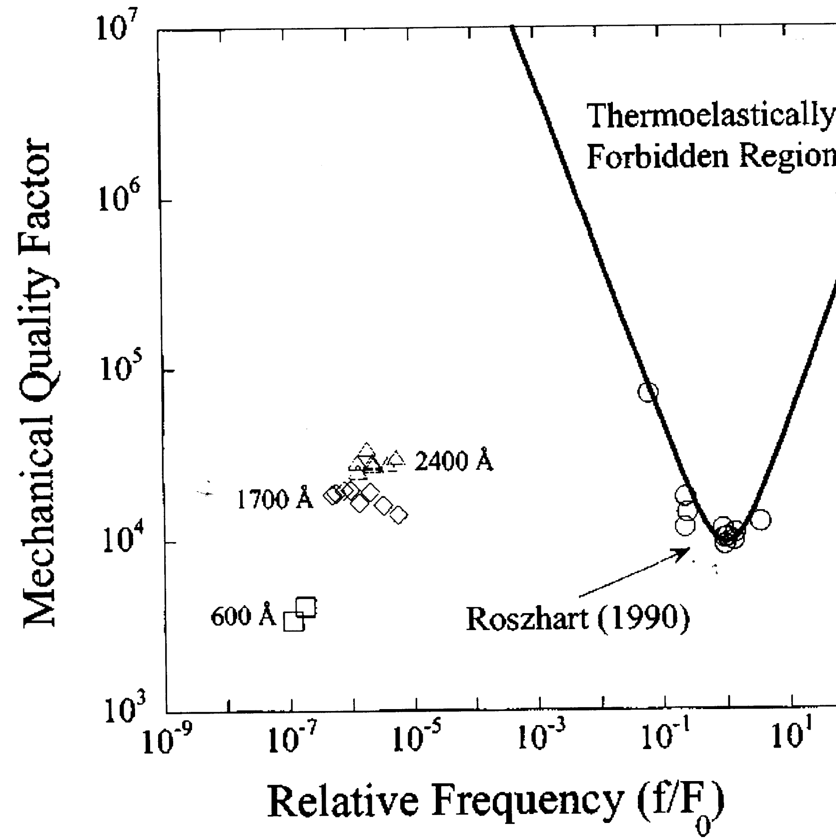
Cantilevers



$$\underbrace{\begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ V_n \end{bmatrix}}_{\mathbf{V}} = \underbrace{\begin{bmatrix} Z_{11} & Z_{12} & \cdot & \cdot & Z_{1n} \\ Z_{21} & Z_{22} & \cdot & \cdot & Z_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ Z_{n1} & Z_{n2} & \cdot & \cdot & Z_{nn} \end{bmatrix}}_{\mathbf{Z}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ \cdot \\ I_n \end{bmatrix}}_{\mathbf{I}},$$

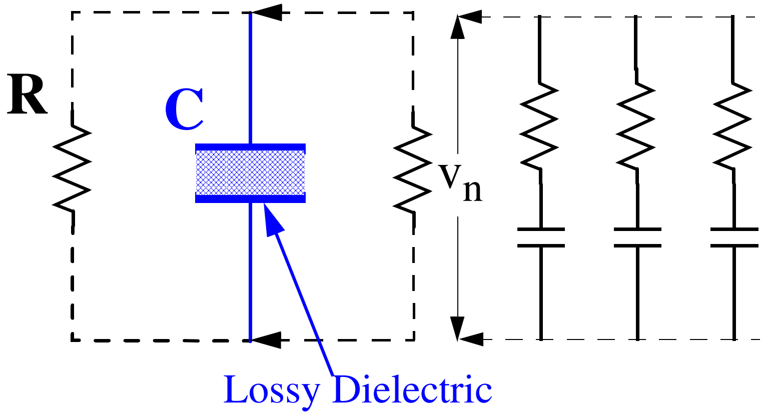
$\underline{189 \times 189}$





- Fused silica optic suspensions:
  $Q \sim 5 \times 10^7$

Lossy Dielectrics and Complex Networks



$$\varepsilon = \varepsilon' + j\varepsilon'' = \varepsilon'\left(1 + j\frac{\varepsilon''}{\varepsilon'}\right)$$

$$\frac{\varepsilon''}{\varepsilon'} = D = \frac{1}{Q} = \frac{g(\omega)}{\omega C}$$

From the Fluctuation-Dissipation Theorem: $\overline{v_n^2} = 4k_B T D \frac{1}{\omega C}$; $\overline{q_n^2} = 4k_B T C \frac{D}{\omega}$

For charge measurement (ns to ms range) $ENC_D \approx k_B T C(2.4D)$

For C = 1 pF at T = 293 K

$\sigma_q = (k_B\ T\ C)^{1/2} = 400$ e rms

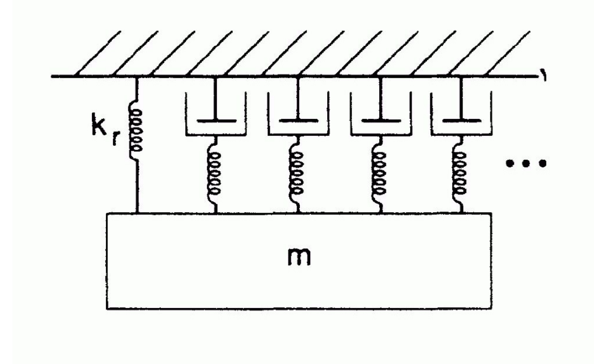
$ENC_a < 5$ e (with an optimized amplifier)

Dielectric Noise Contribution:

$D \approx 5\times10^{-5}$, $ENC_D \approx 4$ e rms

$D \approx 2\times10^{-2}$, $ENC_D \approx 86$ e rms (G10 circuit board)

Complex Spring Constant



$k = k_r(1 + j\varphi)$

k_r = “relaxed” spring constant

$f = 1/Q \approx 10^{-4} - 10^{-8}$

$$v_n^2(\omega) = 4k_B T \frac{R(1 + CDR\omega)}{C^2R^2\omega^2 + (1 + CDR\omega)}$$

$$\sigma_v^2 = \int_0^{f_h} v_n^2(f)\, df \cong \frac{k_B T\left(\pi + \frac{D^2}{\pi RCf_h(1+D^2)} + Log(4\pi^2R^2C^2[1+D^2])\right)}{\pi C(1+D^2)} + \frac{2k_B T Log(f_h)}{\pi C(1+D^2)}$$

$$\sigma_v^2 \approx \frac{k_B T}{C}$$

Concluding Remarks:

1. *Fluctuation-dissipation theorem* with Nyquist's expression for thermal noise is essential for calculation of noise spectra and for detailed information on noise sources.

2. *Equipartition theorem* provides no detailed information, but provides a *check on the integrals* of noise spectra (the total fluctuation).

3. *Transient behavior* of noise in switched capacitor circuits and matrix readout pixel arrays is best understood by means of *Campbell's theorem*, which provides *noise variance vs time*.

4. A complete charge reset and charge transfer by a switch result in $\sigma_q^2 = k_B TC$, independently of the switch ON resistance. This noise can be subtracted only if the *first sample* in the CDS is *before the signal*.

5. Transfer (i.e., direct transport) of charge without switching (as in a CCD) does not result in $k_B TC$ noise. *Reset* of the sense amplifier *does*.

6. In *resonant* sensors and vibration isolators, noise well *below the resonant frequency* is very important. It is inversely proportional to the quality factor Q.

7. *Impulse excitation* of resonant systems smaller than $k_B T$ can be detected by "*single cycle signal processing*".

References

1. H. Nyquist, *"Thermal Agitation of Electric Charge in Conductors"*, Phys. Rev. **32**, 110 (1928).

2. Herbert B. Callen and Richard F. Greene, *"On a Theorem of Irreversible Thermodynamics"*, Phys. Rev. **86**, 701 (1952).

3. Herbert B. Callen and Theodore A. Welton, *"Irreversibility and Generalized Noise"*, Phys. Rev. **83**, 34 (1951).

4. G. E. Uhlenbeck and L. S. Ornstein, *"On The Theory of the Brownian Motion"*, Phys. Rev. **36**, 823 (1930).

5. R. V. Jones and C. W. McCombie, *"Brownian Fluctuations in Galvanometers and Galvanometer Amplifiers"*, Phil. Trans. Roy. Soc. **244**, 205 (1952).

6. Peter R. Saulson, *"Thermal Noise in Mechanical Experiments"*, Phys. Rev. D **42**, 2437 (1990).

7. Thomas B. Gabrielson, *"Mechanical-Thermal Noise in Micromachined Acoustic and Vibration Sensors"*, IEEE Trans. Electron Devices, **40**, 903 (1993).

8. G. G. Yaralioglu and A. Atalar, *"Noise Analysis of Geometrically Complex Mechanical Structures Using the Analogy Between Electrical Circuits and Mechanical Systems"*, Rev. Sci. Instrum. **70,** 2379 (1999).

9. S. Rast, C. Wattinger, U. Gysin, and E. Meyer, *"Dynamics of Damped Cantilevers"*, Rev. Sci. Instrum. **71**, 2772 (2000).

10. Kevin Y. Yasumura, *et al.*, *"Quality Factors in Micron- and Submicron-Thick Cantilevers"*, Journal of Microelectromechanical Systems **9**, 117 (2000).

11. Steven D. Penn, *et al.*, *"High Quality Factor Measured in Fused Silica"*, Rev. Sci. Instrum. **72**, 3671 (2001).

12. D. V. Serebryakov, A. P. Cherkun, B. A. Loginov, and V. S. Letokhov, *"Tuning-Fork-Based Fast Highly Sensitive Surface-Contact Sensor for Atomic Force Microscopy/Near-Field Scanning Optical Microscopy"*, Rev. Sci. Instrum. **73**, 1795 (2002).

13. Robert D. Grober, *et al.*, *"Fundamental Limits to Force Detection Using Quartz Tuning Forks"*, Rev. Sci. Instrum. **71**, 2776 (2000).

14. T. W. Kenny, S. B. Waltman, J. K. Reynolds, and W. J. Kaiser, *"Micromachined Silicon Tunnel Sensor for Motion Detection"*, Appl. Phys. Lett. **58**, 100 (1991).

15. A. N. Cleland and M. L. Roukes, *"A Nanometre-Scale Mechanical Electrometer"*, Nature **392**, 160 (1998).

16. Rahul Sarpeskhar, Tobias Delbrück, and Carver A. Mead, *"White Noise in MOS Transistors and Resistors"*, IEEE Journal of Circuits & Devices **10**, 23 (1993).

17. A. van der Ziel, *"Equivalent Circuit and Equipartition Theorem in Lossy Dielectric or Ferroelectric Capacitors"*, J. Appl. Physic. **44**, 1402 (1973).

18. Doran D. Smith, *"Single Electron Devices"*, Int'l. Journal of High Speed Electronics and Systems **9**, 165 (1998).

19. V. A. Krupenin, *et al.*, *"Noise in Al Single Electron Transistors of Stacked Design"*, J. Appl. Physic. **84**, 3212 (1998).

20. Hui Tian, Boyd Fowler, and Abbas El Gamal, *"Analysis of Temporal Noise in CMOS Photodiode Active Pixel Sensor"*, IEEE Journal of Solid-State Circuits **36**, 92 (2001).

21. C. W. McCombie, *"Fluctuations on a Damped Transmission Line"*, Phys. Rev. **100**, 444 (1955).

22. Daniel G. Maeder, *"Can a Pulse Excitation Smaller than kT be Detected?"*, LBL-523 Preprint (1971).

23. D. G. Lampard, *"The Response of Linear Networks to Suddenly Applied Stationary Random Noise"*, IRE Trans. Circuit Theory, 49 (March 1955).

24. N. R. Campbell (1909), *Proc. Camb. phil. Soc. Math. Phys. Sci.* **15**, 117,310.

25. F.N.H. Robinson, *"Noise and Fluctuations in Electronic Devices and Circuits"*, Clarendon Press, Oxford (1974).

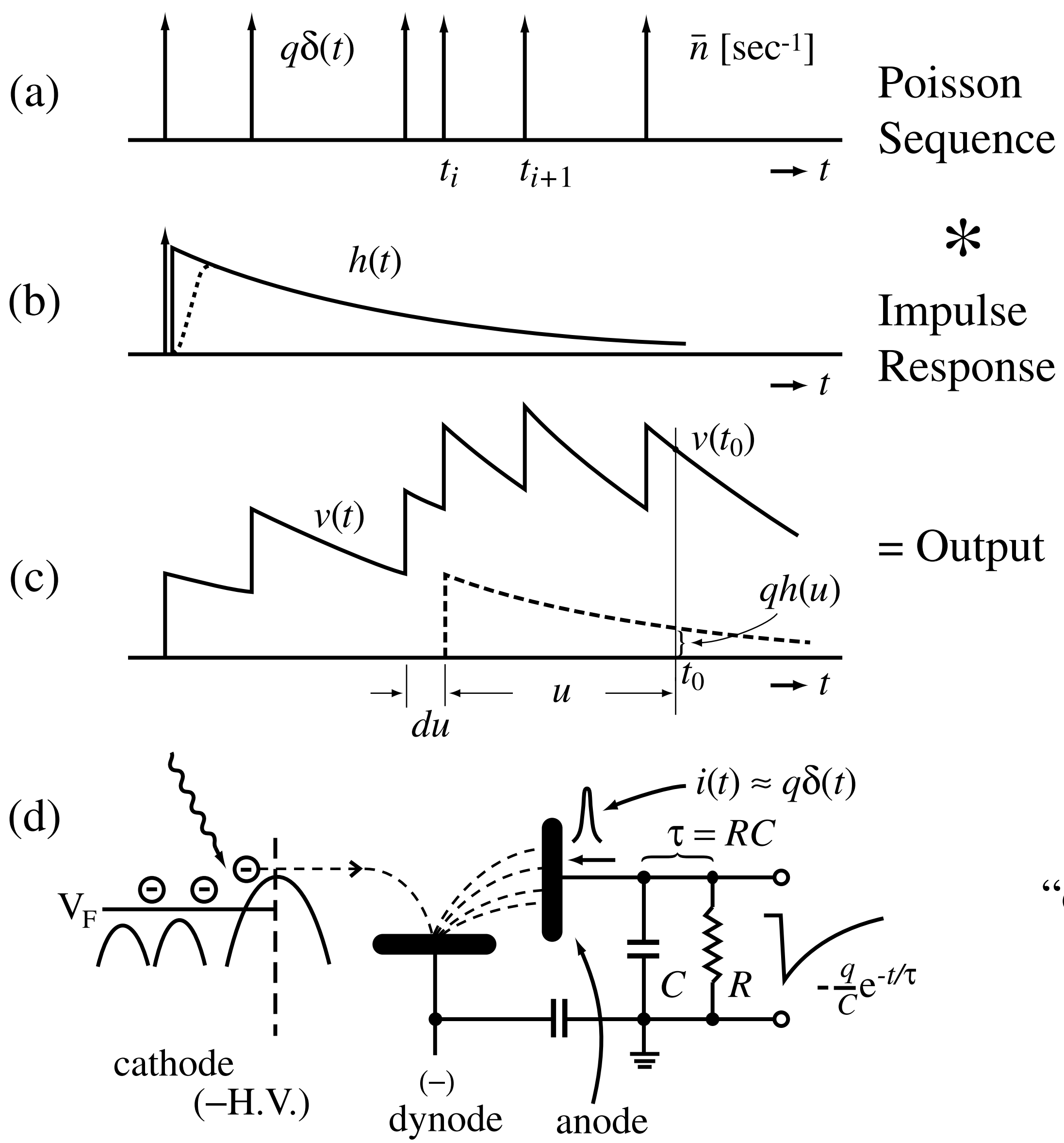
26. Carver Mead, *"Analog VLSI and Neural Systems"*, Addison-Wesley Publishing Company (1989).

27. J.D.E. Beyon and D. R. Lamb, editors, *"Charge-Coupled Devices and Their Applications"*, McGraw-Hill Book Company (UK) Limited (1980).

28. D. F. Barbe, *"Imaging Devices Using the Charge Coupled Concept"*, Proc. IEEE **63**, 38 (1975).

29. R. P. Kraft, *et al.*, *"Soft X-ray Spectroscopy with Sub-Electron Readnoise Charge-Coupled Devices"*, Nucl. Instrum. & Meth. **A361**, 372 (1995).

Shot Noise, "Pileup"



Campbell:

$$\overline{v^2} = \bar{n}\, q^2 \sum_{i=1}^{N} h^2(t_0 - t_i)$$

$$\overline{v^2} = \sigma^2 = \underbrace{\bar{n}\, q^2}_{\text{noise}} \int \underbrace{h^2(u)du}_{\text{system response}}$$

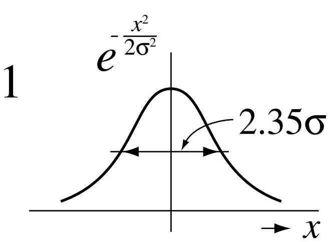
$$\sigma^2 \rightarrow \frac{1}{T_m}\int_0^{T_m} x^2(t)dt$$

Noise Process:

$$\bar{n}q^2 = \bar{n}q \cdot q = \bar{I}_0 \cdot q$$

rate   mean current

"central limit theorem"
→ gaussian for $n\tau \gg 1$



Noise Calculation: *Time Domain and Frequency Domain*

- For time invariant systems impulse response *h(t)* and weighting function *w(t)* are interchangeable for noise calculations.

From Parseval's theorem, *noise variance* can be expressed as:

$$\sigma^2 = \bar{n}\, q^2 \int_{-\infty}^{\infty} h^2(t)\,dt = 2\,\bar{n}\, q^2 \int_{o}^{\infty} |H(\omega)|^2\, df$$

Campbell's theorem

Also: $\sigma^2 = W_o \int_{o}^{\infty} |H(\omega)|^2\, df$

Thus *noise spectral density*: $W_o = 2\,\bar{n}\, q^2 = 2 q_e I_o$ for shot noise
(similarly $W_o = 4kT/R$ for thermal noise)

A.2 HIGH FREQUENCY LIMIT OF NOISE

1. Thermal (Johnson) noise in resistors

$$\overline{v^2} = 4kTR\Delta f \cdot \frac{\frac{hf}{kT}}{\exp\left(\frac{hf}{kT}\right) - 1}$$

$h = 6.63 \times 10^{-34}$ Joules · sec

$k = 1.38 \times 10^{-23}$ Joules

For $\frac{hf}{kT} \ll 1 \quad \longrightarrow \quad \overline{v^2} = 4kTR\Delta f$

For $\frac{hf}{kT} = 1 \quad \longrightarrow \quad f_h \cong \frac{kT}{h}$, at $T = 300°K$

$f_h \approx 6 \times 10^{12}$ Hz

At $T = 0.3\,°K \quad \longrightarrow \quad f_h \approx 6$ GHz

$T = 3 \times 10^{-3}\,°K \quad \longrightarrow \quad f_h \approx 60$ MHz